J. Fluid Mech. (1976), *vol.* 75, *part* 4, *pp.* 609–623
Printed in Great Britain

On the difference between a bounding surface and a material surface

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(Received 30 July 1975)

The ideas of Lagrange, Poisson, Kelvin and Truesdell are reviewed. It is shown that in order for a bounding surface not to be a material surface either $\mathbf{u}.\mathbf{n} = c$ must fail or more than one deformation can be associated with the velocity field. Examples are given.

1. Introduction

A *bounding surface* is any geometric surface which separates a material body from its environment. The material body may be a deformable drop of immiscible oil completely surrounded by water, in which case the bounding surface separates two different materials, or it may be a small parcel of material which is simply part of a larger body, whereby the same type of material lies on both sides of the bounding surface. In any case, it is important to keep in mind that no material can ever pass across a bounding surface. A *material surface* is any surface which always consists of the same material.

One might be tempted to conclude that a bounding surface is, indeed, always a material surface. For example, in an attempt to distinguish the above-mentioned parcel from the rest of the body, we may choose to mark every bit of its matter with an ideal 'dye' that does not diffuse. The dye then reveals the location and shape of the parcel at any instant in time. However, is it necessary to dye the *entire* parcel? Would it not suffice simply to mark only the matter lying on the surface of the parcel? The answer would be yes if the surface bounding the parcel always consists of the same material, which is true as long as the motion of the entire material body is sufficiently smooth. However, the answer is no if the marked surface may tear and fluid which once was located in the interior of the parcel may now form part of the bounding surface. Our aim is to identify conditions under which each situation occurs.

It is commonly thought, especially among those in the field of fluid mechanics, that bounding surfaces are *always* material surfaces. There is a history behind this, and in § 2 the arguments and ideas of Lagrange, Poisson, Kelvin and Truesdell are presented and reviewed critically. In order that a bounding surface should *not* be a material surface, the velocity field of the deforming material must possess a certain feature. This characteristic is derived in § 3. An explicit example is presented which demonstrates the existence of such motions for an incompressible material. In the process of establishing the above, we must make use of the boundary condition $\mathbf{u}.\mathbf{n} = c$, where $\mathbf{u}$ is the velocity of the deforming material at the bounding surface, $\mathbf{n}$ is a unit vector normal to the surface and c is its speed of propagation. For this reason, a short derivation of this well-known boundary condition is given.

2. Historical background

The confusion began with Lagrange (1781), the first person to give a mathematical description of a bounding surface, who *assumed* that bounding surfaces are material surfaces. He realized that this was not entirely true, but his aim was to eliminate the possibility of a single fluid body dividing into two separate pieces. He was aware that this assumption was not sufficient in itself since, even though it eliminated the possibility of the fluid body undergoing a 'fracturing' type of motion, it did not eliminate the possibility of a fluid body 'pinching off'.† Figure 1 gives an interpretation of Lagrange's ideas. Once he had assumed that bounding surfaces are material surfaces, Lagrange proceeded, in a very straightforward manner, to show that a *necessary* characteristic of these surfaces is that

$$\partial F/\partial t + \mathbf{u}.\nabla F = 0, \tag{1}$$

where $F = F(\mathbf{x}, t) = 0$ locates the bounding surface in space (refer to appendix A for details). He also demonstrated the converse by using his own method of characteristics; i.e., if (1) is satisfied, then the surface given by $F(\mathbf{x}, t) = 0$ is a material surface (refer to appendix B for details).

Poisson (1842) objected to Lagrange's assumption that bounding surfaces are material surfaces. He wondered why motions in which a fluid point may move from the interior to the bounding surface must be excluded from consideration. The validity of (1) was thus questioned.

Kelvin (1848) agreed with Poisson's objection and proceeded to restate the definition of a bounding surface: "If a fluid mass be in motion, under any conceivable circumstances, its bounding surface will always be such that there will be no motion of fluid across it." He describes several possible motions in which fluid points originating in the interior are mapped onto the bounding surface. In no case does the entire mass divide into two parts. (Recall Lagrange's motivation for assuming that a bounding surface is a material surface.) Kelvin, motivated to derive a mathematical expression, continues: "To express the fact that every particle of the fluid remains on the same side of the surface, or that there is no *flux* across it, we must find the normal motion of the surface, at any point, in an infinitely small time dt, and equate this to the normal component of the motion of a neighbouring fluid particle during the same time." Equation (1) follows directly (refer to appendix C).

Having begun with a more general definition of a bounding surface, Kelvin

† Lagrange's own words are: "Cette condition paraît en effet nécessaire pour que le fluide ne se divise pas, mais forme toujours une masse continue; cependant nous verrons qu'il y a des cases où elle ne doit pas avoir lieu." The "elle" in the quotation is interpreted as modifying "une masse continue".

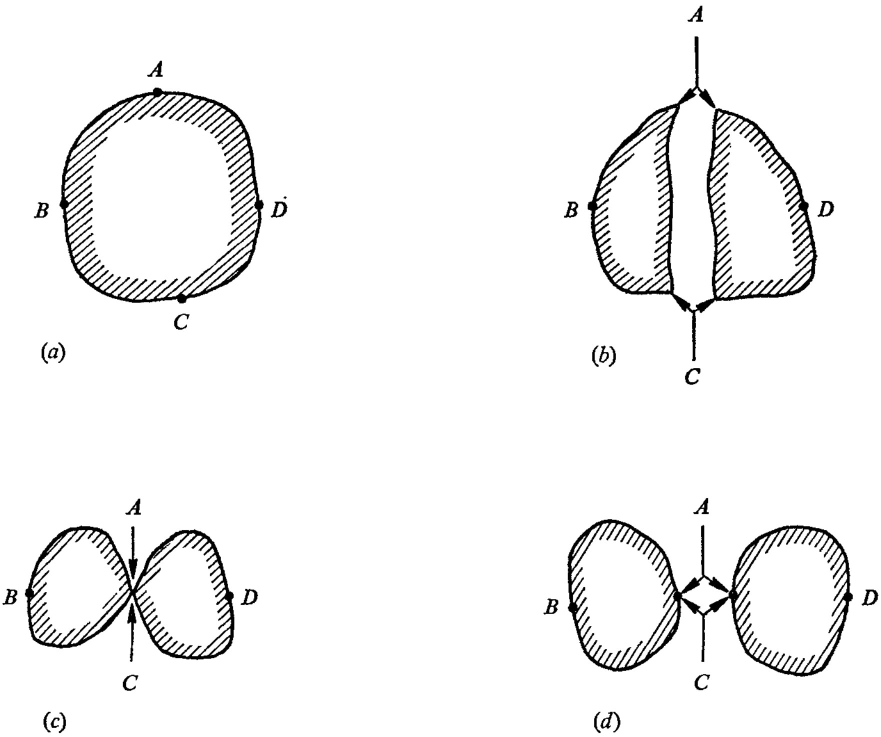

FIGURE 1. (*a*) A, B, C and D are material points on the bounding surface of a continuous mass. (*b*) The mass *fractures*, i.e. 'divides' itself. This type of motion takes material points which were originally in the interior of the mass onto the bounding surface. Lagrange's supposition prohibits this type of motion. However, it is still possible for the mass to divide itself through a 'pinching off' type of motion. This is illustrated in (*c*) and (*d*). (This is the author's interpretation.)

ended by deriving the same equation as Lagrange. Kelvin concludes with: "Poisson has justly remarked that cases may actually occur in which this condition [a bounding surface is a material surface] is violated; but we cannot infer, as he and subsequent authors have done, that the differential equation [equation (1)] is liable to exception in its applications, although we may conclude that the demonstration they have given fails in certain cases."

Kelvin never refers directly to the work of Lagrange; if he had, he would very likely have concluded that *all bounding surfaces are material surfaces.* His reasoning might have been thus: upon a bounding surface $\mathbf{u}.\mathbf{n} = c$ (Kelvin); $\mathbf{u}.\mathbf{n} = c$ implies (1) (Kelvin); (1) implies that the surface $F(\mathbf{x}, t) = 0$ is a material surface (Lagrange). Therefore, in order for a bounding surface *not* to be a material surface either $\mathbf{u}.\mathbf{n} = c$ must not be valid at every point on the surface, or Lagrange's demonstration must break down [it should be noted that Lagrange does *not* state conditions sufficient to ensure that the solutions generated from the characteristic equations give a solution to (1)]. In the first part of § 3 a derivation is given for $\mathbf{u}.\mathbf{n} = c$, and situations in which it is not valid are discussed. In the second part of § 3 it will be shown that, while Lagrange's demonstration is valid for almost all practical flow fields, there does exist a class of motions in which it fails even though the velocity field is continuous and (1) is satisfied. For this class of motions bounding surfaces are *not* material surfaces.

Finally, Truesdell (1951) claims to demonstrate that, if (i) the velocity field of the deforming material is continuous, (ii) the surface $F(\mathbf{x}, t) = 0$ is differentiable, $|\nabla F| \neq 0, \infty$, and satisfies (1), and (iii) the density of the material can never approach zero or infinity, then the surface $F(\mathbf{x}, t) = 0$ must be a material surface. However, Truesdell uses an incomplete definition of a material surface, which invalidates his conclusion (refer to appendix D for details). A counter-example to Truesdell's conclusion is given in § 3.

The main conclusion of the present work is given in § 3. This is that the characteristic which determines whether or not a surface upon which $\mathbf{u}.\mathbf{n} = c$ is a material surface is not the density field but rather the existence of unique trajectories associated with the velocity field, i.e. that there are unique solutions $\mathbf{x} = \mathbf{X}(\mathbf{R}, t)$ of

$$d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{R} = \mathbf{X}(\mathbf{R}, 0)$$

given the velocity field $\mathbf{u}(\mathbf{x}, t)$ associated with the deforming material.

3. Analysis

When are bounding surfaces *not* material surfaces? It was established in § 2, by combining the analyses of Kelvin and Lagrange, that either $\mathbf{u}.\mathbf{n} = c$ must fail at some point on the surface, or else the solution obtained from the characteristic equations must fail to generate a solution to (1).

We shall begin by formalizing Kelvin's definition of a bounding surface for the purpose of deriving sufficient conditions for $\mathbf{u}.\mathbf{n} = c$. Examples will be given in which $\mathbf{u}.\mathbf{n} \neq c$; and the surfaces, while being bounding surfaces, will not be material surfaces. It will then be shown that, if (i) $\mathbf{u}.\mathbf{n} = c$ and (ii) the velocity field associated with the deforming material has a unique motion associated with it, then the bounding surface *is* a material surface. An example will be given of the contrapositive.

In all that will be discussed the only assumption which need be made concerning the nature of the material body is that it is a continuum. A continuum is composed of an infinite number of material points, each point having zero dimension (a material point should not be confused with a molecule). The combined trajectories of all of its material points describes the motion of an entire body. The trajectories are commonly expressed in the form $\mathbf{x} = \mathbf{X}(\mathbf{R}, t)$, where $\mathbf{R} = \mathbf{X}(\mathbf{R}, 0)$. For example, the trajectory of the material point located at $\mathbf{x}_0$ at $t = 0$ is given by $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$. Hence the variable $\mathbf{R}$ is the position vector of any material point in the body at time $t = 0$.

We shall confine outselves to a specific material body which is located within a stationary wall W. The body is divided by a surface S into two disjoint parts $\mathscr{D}_1$ and $\mathscr{D}_2$, whose closures are denoted by $\bar{\mathscr{D}}_1$ and $\bar{\mathscr{D}}_2$, respectively (refer to figure 2). Hence we have that $S = \bar{\mathscr{D}}_1 \cap \bar{\mathscr{D}}_2$.

In order to derive $\mathbf{u}.\mathbf{n} = c$, we begin with a formal statement of Kelvin's definition of a bounding surface.

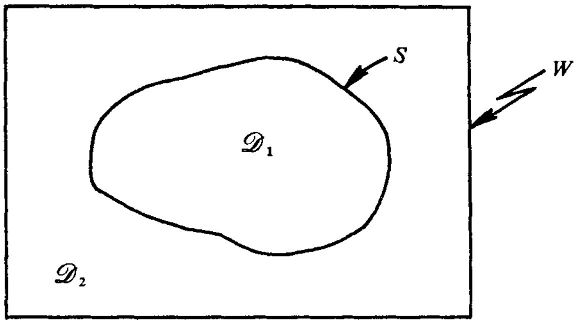

FIGURE 2. The entire material body is bounded by surface W. S divides it into two disjoint parts.

DEFINITION. The surface S is a *bounding surface* if, for all values of time,

$$\mathbf{X}(\mathbf{R}_1, t) \notin \overline{\mathscr{D}}_2 - S, \quad \mathbf{X}(\mathbf{R}_2, t) \notin \overline{\mathscr{D}}_1 - S,$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the positions at $t = 0$ of any material points lying wholly within $\overline{\mathscr{D}}_1$ and $\overline{\mathscr{D}}_2$ respectively.

In other words, if $\overline{\mathscr{D}}_1$ is itself a material body and S is its bounding surface, then no material point originating either from within $\overline{\mathscr{D}}_1$ or outside $\overline{\mathscr{D}}_1$ can ever cross the surface S.

THEOREM 1. If the surface S is a bounding surface possessing a continuous unit normal vector $\mathbf{n}$ and speed of propagation c, and if the velocity field is continuous throughout the entire domain contained within W,† then $\mathbf{u} . \mathbf{n} = c$ upon S.

Proof. If the trajectories of the material points were known, then the velocity field would be given by

$$\bar{\mathbf{u}}(\mathbf{R}, t) \equiv \left(\frac{\partial \mathbf{X}(\mathbf{R}, t)}{\partial t}\right)_{\mathbf{R}}.$$

Sometimes it is more desirable to express the velocity field in terms of the present positions of the material points, $\mathbf{u}(\mathbf{x}, t)$. The two forms are related through

$$\mathbf{u}(\mathbf{x}, t) \equiv \mathbf{u}(\mathbf{X}(\mathbf{R}, t), t) = \bar{\mathbf{u}}(\mathbf{R}, t).$$

However, in this case, it is known only that the velocity field $\mathbf{u}(\mathbf{x}, t)$ is a continuous function. Peano (see Birkhoff & Rota 1969, p. 177) has shown that this is sufficient to conclude that there exists a motion $\mathbf{X}(\mathbf{R}, t)$ which is continuously differentiable in t.‡ This implies that the trajectory of any material point can be represented for sufficiently small t in the following form:

$$\mathbf{X}(\mathbf{R}, t) = \mathbf{X}(\mathbf{R}, 0) + \bar{\mathbf{u}}(\mathbf{R}, 0)\, t + o(t).$$

† In appendix E this is extended to the case in which the materials are permitted to slip along S.

‡ There is only one velocity field associated with any motion, but there may exist more than one motion which can be associated with the same velocity field. The important point is that a function $\mathbf{X}(\mathbf{R}, t)$ exists which is continuous in t (this is necessary for obvious physical reasons).

Any geometric surface can be represented as

$$\mathbf{x} = \mathscr{S}(s^\alpha, t),$$

where $\mathbf{x}$ is the position vector of a point with surface co-ordinates $\{s^\alpha|\alpha = 1, 2\}$. Since the surface S has a continuous unit normal and speed of propagation, it can be represented for sufficiently small t as follows:

$$\mathscr{S}(s^\alpha, t) = \mathscr{S}(s^\alpha, 0) + c\mathbf{n}(s^\alpha, t)\,t + o(t),$$

where the particular surface co-ordinate system used has the property

$$\left(\frac{\partial \mathscr{S}}{\partial t}\right)_{s^\alpha} = c\mathbf{n}.$$

Consider any material point $\mathbf{R}_0$ located on S at $t = 0$, and let s_0^α denote its surface co-ordinates, i.e. $\mathscr{S}(s_0^\alpha, 0) = \mathbf{X}(\mathbf{R}_0, 0)$. At other times these two points need not coincide; in fact

$$\{\mathbf{X}(\mathbf{R}_0, t) - \mathscr{S}(s_0^\alpha, t)\}.\mathbf{n}(s_0^\alpha, t) = \{\bar{\mathbf{u}}(\mathbf{R}_0, 0).\mathbf{n}(s_0^\alpha, 0) - c\}\,t + o(t).$$

If $\mathbf{u}.\mathbf{n} \neq c$, then the left-hand side changes sign as t passes through zero, i.e. the material point $\mathbf{R}_0$ travels across the surface. Therefore, if S is a bounding surface it is necessary that $\mathbf{u}.\mathbf{n} = c$ at all times (the choice of $t = 0$ is arbitrary).

Q.E.D.

Therefore a motion for which $\mathbf{u}.\mathbf{n} \neq c$ on a bounding surface must have a velocity field which is not continuous in the closed domains occupied by the material (refer to appendix E for this extension to theorem 1). For example, consider a material occupying the semi-infinite space $X^1 \leqslant 0$ and undergoing the following motion:

$$(X^1, X^2, X^3) = \begin{cases} (R^1 + U_0 t, R^2, R^3), & t \leqslant -R^1/U_0, \\ (0, R^2, R^3), & t \geqslant -R^1/U_0, \end{cases}$$

where $R^1 \leqslant 0$ and U_0 is a positive constant. The velocity field associated with this motion is

$$(u^1, u^2, u^3) = \begin{cases} (U_0, 0, 0) & X^1 < 0, \\ (\text{undefined}, 0, 0) & X^1 = 0. \end{cases}$$

The X^2, X^3 plane at $X^1 = 0$ is a bounding surface, and its speed of propagation is zero, i.e. $\mathbf{n} = \mathbf{i}$ and $c = 0$; however

$$\mathbf{n}.\lim_{\mathbf{x}\to 0} \mathbf{u}(\mathbf{x}, t) = -U_0.$$

Therefore, for this motion $\mathbf{u}.\mathbf{n} \neq c$ and $X^1 = 0$ is a bounding surface but not a material surface.

The above example has the unappealing characteristic that the mass-density function is unbounded on the plane $X^1 = 0$. However this need not always be the case: Dussan V. & Davis (1974) examined a motion in which $\mathbf{u}.\mathbf{n} \neq c$ at an isolated point, the material may be incompressible, and the bounding surface is still not a material surface.

THEOREM 2. If the surface S possesses a continuous unit normal vector $\mathbf{n}$ and speed of propagation c, if $\mathbf{u}.\mathbf{n} = c$ on S, if the velocity field is continuous throughout the entire domain contained within W†, and if the motion associated with the velocity field is unique, then the surface S must always consist of the same material.‡

Proof. The components of the velocity field lying in the tangent plane of S are given by

$$\mathscr{U}^\alpha = \mathscr{U}^\alpha(s^\beta, t) \equiv \mathbf{u}(\mathscr{S}(s^\beta, t), t).(\partial\mathscr{S}/\partial s^\gamma)\, g^{\gamma\alpha},§$$

where $\{(\partial\mathscr{S}/\partial s^\gamma)\, g^{\gamma\alpha}|\alpha = 1, 2\}$ spans the tangent plane of the surface S, $g^{\gamma\alpha}$ is the inverse of the first fundamental tensor of the surface, $(\partial\mathscr{S}/\partial s^\alpha).(\partial\mathscr{S}/\partial s^\beta)$, and as in theorem 1, $\mathbf{x} = \mathscr{S}(s^\alpha, t)$ gives the position vector of a point on S with surface co-ordinates $\{s^\alpha|a = 1, 2\}$. Note that the functions $\mathscr{U}^\alpha$ are continuous since $\mathbf{u}(\mathbf{x}, t)$ is continuous and S is smooth and moves with a continuous speed of propagation. Peano's existence theorem (Birkhoff & Rota 1969, p. 177) establishes that the differential equation

$$ds^\alpha/dt = \mathscr{U}^\alpha(s^\beta, t), \quad W^\alpha = \xi^\alpha(W^\beta, 0)$$

has a solution which is denoted by

$$s^\alpha = \xi^\alpha(W^\beta, t). \tag{2}$$

In order to establish that (2) can describe trajectories of material points, it must be shown that its velocity field coincides with $\mathbf{u}$ for all points on S. The velocity field associated with (2) is

$$\frac{d\mathbf{x}}{dt} = \left(\frac{\partial\mathscr{S}}{\partial s^\alpha}\right)_t \frac{ds^\alpha}{dt} + \left(\frac{\partial\mathscr{S}}{\partial t}\right)_{s^\beta}. \tag{3}$$

The surface co-ordinate system of theorem 1 is used. Here $(\partial\mathscr{S}/\partial t) = c\mathbf{n}$. Equation (3) can then be expressed as follows:

$$\frac{d\mathbf{x}}{dt} = \left(\frac{\partial\mathscr{S}}{\partial s^\alpha}\right)_t \frac{ds^\alpha}{dt} + c\mathbf{n}. \tag{4}$$

On the other hand, the velocity of the material which instantaneously lies on the surface S can be expressed in the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathscr{S}(s^\beta, t), t) = \mathscr{U}^\alpha(s^\beta, t)\,(\partial\mathscr{S}/\partial s^\alpha)_t + [\mathbf{u}(\mathscr{S}(s^\beta, t), t).\mathbf{n}]\,\mathbf{n}.$$

However, it is assumed that this material on S satisfies $\mathbf{u}.\mathbf{n} = c$. Hence the above can be rewritten as follows:

$$\mathbf{u}(\mathbf{x}, t) = \mathscr{U}^\alpha(s^\beta, t)\,(\partial\mathscr{S}/\partial s^\alpha)_t + c\mathbf{n}. \tag{5}$$

† In appendix F this is extended to the case in which $\mathbf{u}$ is discontinuous across S.

‡ Truesdell & Toupin (1960, p. 509) assert that "the bounding surface of every body in a topological motion is a material surface". This is different: theorem 2 assumes that $\mathbf{u}.\mathbf{n} = c$ and not that S is a bounding surface.

§ All Greek letters can take on the values 1 and 2. Repeated indices imply summation.

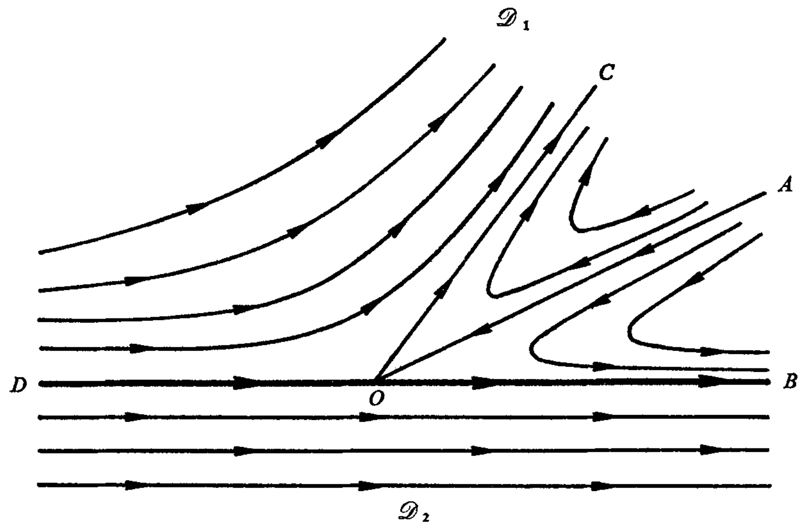

FIGURE 3. The solid lines are streamlines and the arrows indicate the direction of flow. The angles AOB and COB are 30° and 65°, respectively. The line DB is the surface S.

Upon combining (4) and (5), it follows that

$$ds^\alpha/dt = \mathscr{U}^\alpha(s^\alpha, t) \quad \text{implies} \quad d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t) \quad \text{for all points on} \quad S.$$

In other words, the velocity associated with (2) is identical to the value of $\mathbf{u}$ on S. This, along with the assumption that the motion associated with $\mathbf{u}$ is unique, implies that $\mathbf{x} = \mathscr{S}(\xi^\alpha(W^\beta, t), t)$ are trajectories of material points. Hence S is a material surface. Q.E.D.

To illustrate the contrapositive, two flow fields are examined. In both cases $\mathbf{u}.\mathbf{n} = c$ and S is not a material surface; however, in one case S is a bounding surface, while in the other it is not. The essential characteristic, in both cases, is that more than one motion can be associated with each velocity field.

In the first example, the flow is two-dimensional and the velocity field is given by

$$u(r,\phi) = \begin{cases} -r^{\frac{1}{2}}\{-7{\cdot}033\cos\frac{1}{2}(\phi-\frac{1}{6}\pi)+7{\cdot}386\cos\frac{3}{2}(\phi-\frac{1}{6}\pi) \\ \qquad +0{\cdot}306\sin\frac{1}{2}(\phi-\frac{1}{6}\pi)-0{\cdot}919\sin\frac{3}{2}(\phi-\frac{1}{6}\pi)\}, & 0\leqslant\phi\leqslant\frac{1}{6}\pi, \\ -r^{\frac{1}{2}}\{0{\cdot}315\cos\frac{1}{2}(\phi-\frac{1}{6}\pi)+0{\cdot}038\cos\frac{3}{2}(\phi-\frac{1}{6}\pi) \\ \qquad +0{\cdot}306\sin\frac{1}{2}(\phi-\frac{1}{6}\pi)-0{\cdot}919\sin\frac{3}{2}(\phi-\frac{1}{6}\pi)\}, & \frac{1}{6}\pi\leqslant\phi\leqslant\pi, \\ \cos\phi, & \pi\leqslant\phi\leqslant 2\pi, \end{cases}$$

$$v(r,\phi) = \begin{cases} r^{\frac{1}{2}}\{-21{\cdot}099\sin\frac{1}{2}(\phi-\frac{1}{6}\pi)+7{\cdot}386\sin\frac{3}{2}(\phi-\frac{1}{6}\pi) \\ \qquad -0{\cdot}919\cos\frac{1}{2}(\phi-\frac{1}{6}\pi)+0{\cdot}919\cos\frac{3}{2}(\phi-\frac{1}{6}\pi)\}, & 0\leqslant\phi\leqslant\frac{1}{6}\pi, \\ r^{\frac{1}{2}}\{0{\cdot}945\sin\frac{1}{2}(\phi-\frac{1}{6}\pi)+0{\cdot}038\sin\frac{3}{2}(\phi-\frac{1}{6}\pi) \\ \qquad -0{\cdot}919\cos\frac{1}{2}(\phi-\frac{1}{6}\pi)+0{\cdot}919\cos\frac{3}{2}(\phi-\frac{1}{6}\pi)\}, & \frac{1}{6}\pi\leqslant\phi\leqslant\pi, \\ \sin\phi, & \pi\leqslant\phi\leqslant 2\pi, \end{cases}$$

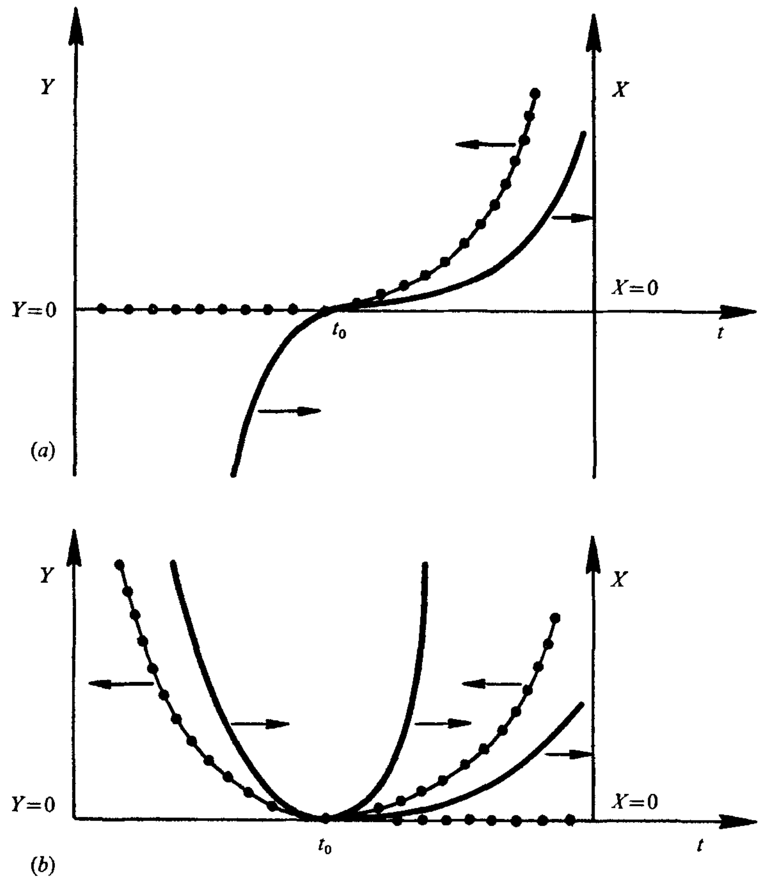

FIGURE 4. The trajectories of two material points *belonging to* $\overline{\mathscr{D}}_1$ and arriving at O at $t = t_0$. The curves in (*a*) describe the trajectory of the material point located at $\{r = 1;\ \phi = \pi\}$ at $t = t_0 - 2$. The dots serve only to distinguish the $Y(t)$ from the $X(t)$ curve. The curves in (*b*) describe the trajectory of the material point located at $\{r = 0{\cdot}595;\ \phi = 30°\}$ at $t = t_0 - 2$. At $t = t_0$ the material point splits in two; thus, for $t > t_0$ there are two sets of curves for both $Y(t)$ and $X(t)$. The curves given in (*a*) and (*b*) are described by (6) and (7) respectively.

where $\mathbf{u}(r, \phi) = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\phi}}$; refer to figure 3. The surface S is given by $\{\phi = 0, \pi;\ r \geqslant 0\}$. The material above S, i.e. $0 \leqslant \phi \leqslant \pi$, is denoted by $\overline{\mathscr{D}}_1$ and the material below S is denoted by $\overline{\mathscr{D}}_2$. There are many motions which give rise to this velocity field; we are concerned with the specific motion which possesses the characteristics illustrated in figure 4. The equations for the trajectories given in this figure are as follows:

$$X = \begin{cases} -(t_0 - t)^{\frac{3}{2}}, & t \leqslant t_0, \\ 0{\cdot}167(t - t_0)^{\frac{3}{2}}, & t \geqslant t_0, \end{cases} \tag{6a}$$

$$Y = \begin{cases} 0, & t \leqslant t_0, \\ 0{\cdot}359(t - t_0)^{\frac{3}{2}}, & t \geqslant t_0, \end{cases} \tag{6b}$$

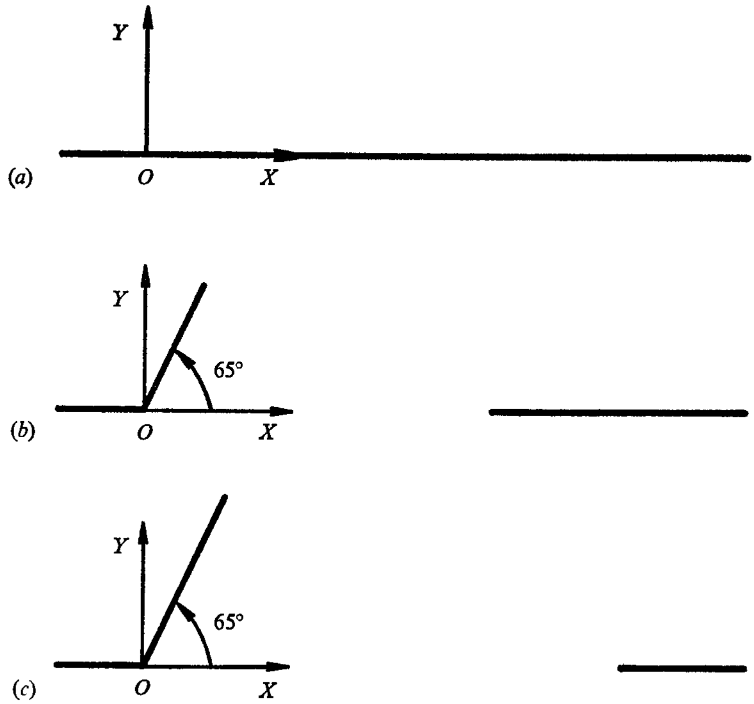

FIGURE 5. The surface DB in figure 3 is not a material surface with respect to $\bar{\mathscr{D}}_1$. The curves in (a), (b) and (c) are the locations of the material belonging to $\bar{\mathscr{D}}_1$ at times $t = 0$, 1 and 2; this location coincides with DB at $t = 0$. The surface DB is not a $\bar{\mathscr{D}}_1$ material surface because curves (b) and (c) are not identical to (a).

and

$$X = \left\{\begin{array}{ll} 0{\cdot}364(t_0-t)^{\frac{3}{2}}, & t \leqslant t_0, \\ 0{\cdot}167(t-t_0)^{\frac{3}{2}}, & t \geqslant t_0, \\ (t-t_0)^{\frac{3}{2}}, & t \geqslant t_0, \end{array}\right\} \tag{7a}$$

$$Y = \left\{\begin{array}{ll} 0{\cdot}210(t_0-t)^{\frac{3}{2}}, & t \leqslant t_0, \\ 0{\cdot}359(t-t_0)^{\frac{3}{2}}, & t \geqslant t_0, \\ 0, & t \geqslant t_0. \end{array}\right\} \tag{7b}$$

That is to say, the material points belonging to $\bar{\mathscr{D}}_1$ and lying on $\{\phi = \pi; r > 0\}$ eventually travel to the ray $\{\phi = 65°; r > 0\}$, and the material points lying on $\{\phi = 0; r > 0\}$ come from $\{\phi = 30°; r > 0\}$. On the other hand, the trajectories of the material points on S belonging to $\bar{\mathscr{D}}_2$ are given by

$$X = t-t_0, \quad Y = 0.$$

It is easy to verify that the material is incompressible, that $\mathbf{u}.\mathbf{n} = c$, that S is a bounding surface, and that it is a material surface with respect to $\bar{\mathscr{D}}_2$ but *not* with respect to $\bar{\mathscr{D}}_1$. To emphasize the last point, the locations at subsequent times of the material surface belonging to $\bar{\mathscr{D}}_1$ and coinciding with S at $t = 0$ are shown in figure 5.

For the last example, consider the unidirectional motion given by†

$$(X^1, X^2, X^3) = ([\tfrac{1}{3}t + (R^1)^{\frac{1}{3}}]^3, R^2, R^3).$$

The velocity field associated with this motion is

$$(u^1, u^2, u^3) = ((X^1)^{\frac{2}{3}}, 0, 0).$$

The other trajectories associated with the above velocity field are

$$(X^1, X^2, X^3) = \begin{cases} [(\tfrac{1}{3}t + (R^1)^{\frac{1}{3}}]^3, R^2, R^3), & -\infty < t \leqslant -3(R^1)^{\frac{1}{3}}, \\ (0, R^2, R^3), & -3(R^1)^{\frac{1}{3}} \leqslant t \leqslant K, \\ ([\tfrac{1}{3}(t-K) + (R^1)^{\frac{1}{3}}]^3, R^2, R^3), & K \leqslant t < \infty, \end{cases}$$

where K is any constant larger than $-3(R^1)^{\frac{1}{3}}$. The surface S is given by $X^1 = 0$. In this case $\mathbf{u}.\mathbf{n} = c$; however S is neither a bounding nor a material surface.

4. Discussion

It has been shown that, if a bounding surface is not a material surface, then either $\mathbf{u}.\mathbf{n} = c$ must be invalid, or else the ordinary differential system

$$d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$$

must have more than one solution for a given set of initial conditions. It is *erroneous* to conclude that if $\mathbf{u}.\mathbf{n} = c$ on S then S is a bounding surface. It is true that one solution (theorem 2) corresponds to S being a material surface (hence a bounding surface), but, if there exists more than one solution to $d\mathbf{x}/dt = \mathbf{u}$ this might *not* be the one associated with the motion of the deforming material. The fact that having $\mathbf{u}.\mathbf{n} = c$ on S implies zero 'flux' across S, by the very definition of the word, does not necessarily imply that material cannot cross S. Higher terms are needed in the Taylor series expansion of the trajectory of material points instantaneously on S. In the last example in § 3, the trajectory of a material point instantaneously on $X^1 = 0$ at $t = 0$ is given by

$$X^1 = \tfrac{1}{27}t^3.$$

The material point leaves the surface even though $\mathbf{u}.\mathbf{n} = 0$, and $(d\mathbf{u}/dt).\mathbf{n} = 0$ on the stationary surface $X^1 = 0$.

When is the solution to $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$, $\mathbf{R} = \mathbf{X}(\mathbf{R}, 0)$ unique? It is well known that it is sufficient for $\mathbf{u}$ to be Lipschitz continuous (Ince 1956, p. 62); however, this condition is by no means necessary (an example can be found in Ince 1956, p. 67). Lamb (1932, p. 7) demonstrates by expanding the motion of the material locally about S that, if $\mathbf{u}.\mathbf{n} = c$ and $\mathbf{u}$ is Lipschitz continuous, then S is a material surface. However, it is shown in theorem 2 that the existence of a unique motion is a more general criterion.

The author gratefully acknowledges Stephen H. Davis, who has been a constant source of stimulating conversation, Fred Kafka for his calculations in two examples, and the National Science Foundation, Engineering Mechanics Program, for its support through grants GK31794, GK 40714 and ENG75-10297.

† See Truesdell (1951).

Appendix A

Lagrange's derivation of the expression which is commonly referred to as the kinematic boundary condition,

$$\partial F/\partial t + \mathbf{u}.\nabla F = 0,$$

is quite direct. He begins by following the trajectory of a material point that is located on the surface at position $\mathbf{x}$ at time t, i.e. $F(\mathbf{x}, t) = 0$. Since he assumes that a bounding surface is a material surface, it follows that $\mathbf{x} + \mathbf{u}\Delta t$ must lie on the bounding surface at $t + \Delta t$, i.e. $F(\mathbf{x} + \mathbf{u}\Delta t, t + \Delta t) = 0$. If it is also permissible to write

$$F(\mathbf{x} + \mathbf{u}\Delta t, t + \Delta t) = F(\mathbf{x}, t) + \nabla F.\mathbf{u}\Delta t + (\partial F/\partial t)\,\Delta t + o(\Delta t),$$

then by making the appropriate substitutions and taking the limit as $\Delta t \to 0$ we get the above-mentioned equation.

Appendix B

Lagrange demonstrates by his method of characteristics (1779) that if $F = F(\mathbf{x}, t)$ is a function satisfying (1) then the surfaces given by $F(\mathbf{x}, t) =$ constant are fluid material surfaces. The equations for the characteristics are

$$d\mathbf{x}/dt = \mathbf{u}, \quad dF/dt = 0. \tag{B 1}$$

These equations are solved assuming the following initial condition:

$$h = f(\mathbf{R}),$$

where $\mathbf{R}$ and h are the initial values (say at $t = 0$) of $\mathbf{x}$ and F respectively, and f is any arbitrary function of $\mathbf{R}$. The solutions to (1) can be expressed in the following form:

$$\mathbf{x} = \mathbf{X}(\mathbf{R}, t), \quad F = h.$$

In physical terms the function $\mathbf{X}$, upon holding $\mathbf{R}$ fixed, describes the trajectory of the fluid material point located at $\mathbf{R}$ at $t = 0$. The solution $F = F(\mathbf{x}, t)$ is obtained by making the following series of substitutions (from left to right):

$$F = h = f(\zeta(\mathbf{x}, t)) = F(\mathbf{x}, t),$$

where ζ is the inverse function of $\mathbf{X}$, i.e. $\mathbf{R} \equiv \zeta(\mathbf{X}(\mathbf{R}, t), t)$ for all values of t.

This demonstrates that any surface given by $F(\mathbf{x}, t) =$ constant, in particular the surface $F(\mathbf{x}, t) = 0$, is a fluid material surface.

Appendix C

Kelvin states that his definition of a bounding surface implies

$$\mathbf{u}.\mathbf{n} = c, \tag{C 1}$$

where $c\mathbf{n}\,dt$ is the "normal motion of the surface, at any point, in an infinitely small time" and $\mathbf{u}.\mathbf{n}\,dt$ is the "normal motion of a neighbouring fluid point during the same time". If we choose to describe the bounding surface by $F(\mathbf{x}, t) = 0$ then

$$\mathbf{n} = \nabla F/|\nabla F|, \quad c = -(\partial F/\partial t)/|\nabla F|.$$

Upon substituting the above into (C 1) we get (1).

Appendix D

Truesdell's analysis proceeds as follows. He examines the position of the surface in both the present space, $\mathbf{x}$, and in some reference space $\mathbf{R}$. These two spaces are related through the motion of the material: $\mathbf{x} = \mathbf{X}(\mathbf{R}, t)$, where $\mathbf{R} = \mathbf{X}(\mathbf{R}, 0)$. The image of the surface $F(\mathbf{x}, t) = 0$ in the reference space is given by $\bar{F}(\mathbf{R}, t) = 0$,† where

$$\bar{F}(\mathbf{R}, t) \equiv F(\mathbf{X}(\mathbf{R}, t), t) = F(\mathbf{x}, t).$$

The normal speed of the surface in the present space, c, is defined in § 2. The normal speed of the surface in the reference space, c_0, is given by

$$c_0 \equiv -\frac{\partial \bar{F}}{\partial t}\Bigg/\left\{\left(\frac{\partial \bar{F}}{\partial R^1}\right)^2 + \left(\frac{\partial \bar{F}}{\partial R^2}\right)^2 + \left(\frac{\partial \bar{F}}{\partial R^3}\right)^2\right\}^{\frac{1}{2}}.$$

Finally, the normal speed v_n of the material points instantaneously on the surface is given by

$$v_n \equiv \mathbf{u}.\mathbf{n}.$$

Combining the above, along with the definitions of $\mathbf{n}$ given in § 2, we get

$$\left(\frac{\partial \bar{F}}{\partial t}\right)_{\mathbf{R}} = \left(\frac{\partial F}{\partial t}\right)_{\mathbf{x}} + \mathbf{u}.\nabla F = (v_n - c)\,|\nabla F| = c_0\left\{\left(\frac{\partial \bar{F}}{\partial R^1}\right)^2 + \left(\frac{\partial \bar{F}}{\partial R^2}\right)^2 + \left(\frac{\partial \bar{F}}{\partial R^3}\right)^2\right\}^{\frac{1}{2}} \quad \text{(D 1)}$$

for all points $\mathbf{x}$ on $F(\mathbf{x}, t) = 0$ and all points $\mathbf{R}$ on $\bar{F}(\mathbf{R}, t) = 0$.

Conclusions can now be drawn. If the *necessary* condition $v_n = c$ for $F(\mathbf{x}, t) = 0$ to be a bounding surface is substituted into (D 1), then Kelvin's result follows; i.e. $\mathbf{u}.\mathbf{n} = c$ implies (1). Next, Truesdell examines the converse. If (1) is satisfied and

$$\left(\frac{\partial \bar{F}}{\partial R^1}\right)^2 + \left(\frac{\partial \bar{F}}{\partial R^2}\right)^2 + \left(\frac{\partial \bar{F}}{\partial R^3}\right)^2 \neq 0 \quad \text{(D 2)}$$

then (D 1) gives $c_0 = 0$. What does this imply? We know that in order for $\bar{F}(\mathbf{R}, t) = 0$ to be a material surface it is necessary and sufficient for the surface $\bar{F}(\mathbf{R}, t) = 0$ to be given by $\bar{F}(\mathbf{R}, 0) = 0$ for all t. However it is *not true in general* that

$$\{\bar{F}(\mathbf{R}, t) = \bar{F}(\mathbf{R}, 0) = 0 \quad \text{are identical surfaces}\} \quad \text{is equivalent to} \quad c_0 = 0.‡$$

† $F(\mathbf{x}, t)$ and $F(\mathbf{R}, t)$ correspond to Truesdell's $f(\mathbf{x}, t)$ and $F(\mathbf{R}, t_0, t)$.

‡ The surface $\bar{F}(\mathbf{R}, t) = 0$ may 'extend itself' at its perimeter, like a 'snake'; i.e. $c_0 = 0$ everywhere except at the perimeter, where it is not defined. This is illustrated in figure 5: note that the deforming material is incompressible.

If we restrict ourselves to those cases in which the above statement is valid, then we can draw the conclusion

$$\partial F/\partial t+\mathbf{u}.\nabla F = 0 \quad \text{implies that} \quad F(\mathbf{x},t) = 0 \quad \text{is a material surface,}$$

provided, of course, that (D 2) is valid, which Tiuesdell shows to be equivalent to the density not approaching zero or infinity.

Appendix E

The proof given for theorem 1 fails when S is a surface upon which the materials may slip. Nevertheless, the theorem is still valid if $\mathbf{u}(\mathbf{x},t)$ is continuous in the closed domains $\overline{\mathscr{D}}_1$ and $\overline{\mathscr{D}}_2$.† We begin by focusing our attention on $\overline{\mathscr{D}}_1$. We want to demonstrate that if S is a bounding surface then $\mathbf{u}_1.\mathbf{n} = c$, where

$$\mathbf{u}_1(\mathbf{x},t) \equiv \lim_{\substack{\mathbf{x}_1\to\mathbf{x}\\ \mathbf{x}_1\in\mathscr{D}_1}} \mathbf{u}(\mathbf{x}_1,t) \quad \text{for} \quad \mathbf{x}\in S;$$

i.e. $\mathbf{u}_1$ is the velocity associated with $\overline{\mathscr{D}}_1$.

Proof. Since it is assumed that $\overline{\mathscr{D}}_1$ is a material body, i.e. no material point originating in $\overline{\mathscr{D}}_1$ can cross S, the velocity field in $\overline{\mathscr{D}}_2$ can be replaced by one which is continuous with $\mathbf{u}_1$ across S without affecting the trajectory of any material point contained within $\overline{\mathscr{D}}_1$. The proof given for theorem 1 in § 3 is used with respect to this new velocity field, and the conclusion, $\mathbf{u}_1.\mathbf{n} = c$ on S, follows immediately. Q.E.D.

In a similar manner, it follows that $\mathbf{u}_2.\mathbf{n} = c$ on S.

Appendix F

The materials in domains $\overline{\mathscr{D}}_1$ and $\overline{\mathscr{D}}_2$ must be examined separately, as is done in appendix E. Domain $\overline{\mathscr{D}}_1$ is considered first.

THEOREM 2.1. If the surface S possesses a continuous unit normal vector $\mathbf{n}$ and speed of propagation c, if $\mathbf{u}_1.\mathbf{n} = c$ on S, and if a velocity field can be constructed in $\overline{\mathscr{D}}_2$ such that the entire velocity field is continuous within W and there is a unique motion associated with it, then no material point originating on S can ever travel to the interior of $\overline{\mathscr{D}}_1$.

Proof. It is immaterial if more than one velocity field can be constructed in $\overline{\mathscr{D}}_2$; it is only necessary that one exists. Theorem 2, with this amended velocity field, gives that S is a material surface. Since the trajectories of the material points belonging to $\overline{\mathscr{D}}_1$ come from the velocity field of $\overline{\mathscr{D}}_1$ and not $\overline{\mathscr{D}}_2$, it can be concluded that no material point originating on S can be mapped into the interior of $\overline{\mathscr{D}}_1$.

However, this does not preclude the possibility of a material point originating on S travelling into the interior of $\overline{\mathscr{D}}_2$. This dual identity of material points at S is a consequence of modelling material bodies as closed sets.

† The velocity field is double valued on S because material bodies are closed sets.

THEOREM 2.2. If the surface S possesses a continuous unit normal vector $\mathbf{n}$ and speed of propagation c, if $\mathbf{u}_2 . \mathbf{n} = c$ on S, and if a velocity field can be constructed in $\overline{\mathscr{D}}_1$ such that the entire velocity field within W is continuous and there is a unique motion associated with it, then no material point originating on S can ever travel to the interior of $\overline{\mathscr{D}}_2$.

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